Journal of Engineering Physics and Thermophysics, Vol. 81, No. 6,

NUMERICAL-ANALYTICAL METHOD OF SOLUTION OF A NONLINEAR UNSTEADY HEAT-CONDUCTION EQUATION

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UDC 536.2.083

A method of solution of a one-dimensional nonlinear unsteady heat-conduction equation has been proposed. The use of the method of Green's functions made it possible to transform the resulting equation to a nonlinear Volterra integral equation of the second kind for temperature, which is solved by the quadratic-form method. A system of recurrence relations, which is solved numerically, has been obtained. The influence of the nonlinearity on the temperature profiles has been analyzed. A comparison to the numerical finite-element method has shown that the numerical-analytical technique allows a reduction of more than 10^3 times in the calculation time.

Introduction. The nonlinearity of a heat-conduction equation is related to either the temperature dependence of the thermophysical properties of the material under study or the complex boundary conditions, e.g., the necessity of allowing for the radiation from the surface. Such problems can be solved only numerically, usually by the finite-element method, which is time-consuming as far as calculations are concerned. In this work, we propose a method for solving of a one-dimensional nonlinear unsteady heat-conduction equation that is transformed so that the method of Green's functions usually used for linear problems can be used [1, 2]. This technique allows a considerable reduction in the calculation time.

Formulation and Solution of the Problem. Let us consider the heating of a half-space by the convective heat flux $q_c = \alpha(T_g - T_0)$ on the source side of the surface. The surface radiates the heat flux $q_r = \varepsilon_r\sigma T_0^4$. The thermal conductivity and the heat capacity of the material are functions of the temperature: $\lambda = \lambda(T)$ and $c = c(T)$, whereas the density ρ is constant. Thus, it is necessary to solve the heat-conduction equation

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) \tag{1}$$

with the initial condition $T(x)\big|_{t=0} = T_{in}$ and the boundary conditions

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = -\frac{1}{\lambda} q_c(T_0)\,;\quad q_c(T_0) = \alpha(T_g - T_0) - \varepsilon_r\sigma T_0^4\,;\quad T(x)\big|_{x\to\infty} = T_{in}\,. \tag{2}$$

Let us separate the nonlinear term in Eq. (1) into an individual term

$$\frac{\partial T}{\partial t} = a_{in}\frac{\partial^2 T}{\partial x^2} + F(T)\,,\quad F(T) = \frac{1}{\rho c}\frac{\partial \lambda}{\partial T}\left(\frac{\partial T}{\partial x}\right)^2 + (a - a_{in})\frac{\partial^2 T}{\partial x^2}\,. \tag{3}$$

As a result, Eq. (1) is transformed to a linear one with virtual internal heat sources.

In solving Eq. (3), we use the method of Green's functions, according to which

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$$T(x,t)=\int_0^{\infty}(GT)_{\tau=0}\,d\xi-a_{\rm in}\int_0^t\left(G\frac{\partial T}{\partial \xi}\right)_{\xi=0}d\tau+a_{\rm in}\int_0^t\left(T\frac{\partial G}{\partial \xi}\right)_{\xi=0}d\tau+\int_0^t\int_0^{\infty}(FG)_{\xi,\tau}\,d\xi d\tau\,. \tag{4}$$

The Green's function $G(x, \xi, t, \tau)$ must satisfy the following conditions:

$$\frac{\partial G}{\partial \tau}=-a_{\rm in}\frac{\partial^2 G}{\partial \xi^2}\,,\quad \int_0^{\infty}(GT)_{\tau\to t}=T(x,t)\,. \tag{5}$$

Also, it is necessary that the boundary condition

$$\left.\frac{\partial G}{\partial \xi}\right|_{\xi=0}=0 \tag{6}$$

be fulfilled. According to [3], the above conditions are satisfied by the function

$$G(x,\xi,t,\tau)=G_0(x-\xi,t-\tau)+G_0(x+\xi,t-\tau)\,,\quad G_0(x,t)=\frac{1}{2\sqrt{\pi a_{\rm in}t}}\exp\left[-\frac{x^2}{4a_{\rm in}t}\right]. \tag{7}$$

Substituting (7) into (4) and taking (2) into account, we obtain an expression for determination of $T(x, t)$:

$$T(x,t)=T_{\rm in}-a_{\rm in}\int_0^t G\,|_{\xi=0}\,q_{\rm c}(T_0)\,d\tau+\int_0^t\int_0^{\infty}[GF(T)]_{\xi,\tau}\,d\xi d\tau\,. \tag{8}$$

With the aim of simplifying transformations and calculations we pass to dimensionless variables

$$\bar t=\frac{t}{t_*}\,;\ \bar\tau=\frac{\tau}{t_*}\,;\ \bar x=\frac{x}{x_*}\,;\ \bar\xi=\frac{\xi}{x_*}\,;\ \theta=\frac{T}{T_{\rm in}}\,;\ \bar\lambda=\frac{\lambda}{\lambda_{\rm in}}\,;\ \bar c=\frac{c}{c_{\rm in}}\,;\ \bar q_{\rm c}=\frac{q_{\rm c}x_*}{\lambda_{\rm in}T_{\rm in}}\,;\ \bar F=\frac{Fx_*^2}{\lambda_{\rm in}T_{\rm in}}\,;\ x_*=\frac{\lambda_{\rm in}}{\alpha}\,;\ t_*=\frac{\pi x_*^2}{a_{\rm in}}\,.$$

As a result, relations (2), (3), and 8) will take the following form:

$$\bar q_{\rm c}(\theta_0)=\frac{1}{\bar\lambda(\theta_0)}\left[\theta_0-\theta_{\rm g}+\beta\theta_0^4\right],\quad \beta=\frac{\varepsilon_{\rm r}\sigma T_{\rm in}^3}{\alpha}\,; \tag{9}$$

$$\bar F(\theta)=\frac{1}{\bar c}\frac{\partial\bar\lambda}{\partial\theta}\left(\frac{\partial\theta}{\partial\bar x}\right)^2+\left(\frac{\bar\lambda}{\bar c}-1\right)\frac{\partial^2\theta}{\partial\bar x^2}\,; \tag{10}$$

$$\theta(\bar x,\bar t)=1-\int_0^{\bar t}\frac{\bar q_{\rm c}(\theta_0)}{\sqrt{\bar t-\bar\tau}}\exp\left[-\frac{\bar x^2}{4\pi(\bar t-\bar\tau)}\right]d\bar\tau+\frac{1}{2}\int_0^{\bar t}\int_0^{\infty}\bar F(\theta)\left\{\exp\left[-\frac{(\bar x-\bar\xi)^2}{4\pi(\bar t-\bar\tau)}\right]+\exp\left[-\frac{(\bar x+\bar\xi)^2}{4\pi(\bar t-\bar\tau)}\right]\right\}\frac{d\bar\xi d\bar\tau}{\sqrt{\bar t-\bar\tau}}\,. \tag{11}$$

On the surface ($\bar x$ = 0), we have

$$\theta_0(\bar t)=1-\int_0^{\bar t}\frac{\bar q_{\rm c}(\theta_0)}{\sqrt{\bar t-\bar\tau}}\,d\bar\tau+\int_0^{\bar t}\int_0^{\infty}\bar F(\theta)\exp\left[-\frac{\bar\xi^2}{4\pi(\bar t-\bar\tau)}\right]\frac{d\bar\xi d\bar\tau}{\sqrt{\bar t-\bar\tau}}\,. \tag{12}$$

Equation (12) is a nonlinear Volterra integral equation of the second kind for θ_0. It can be solved by the quadratic-form method by replacing the integral in (12) by the finite sum of terms [4]. For this purpose the time $\bar{t}$ is subdivided into $(n-1)$ intervals; $\bar{t}_1 = 0$, $\bar{t}_1$, $\bar{t}_2$, ..., $\bar{t}_j$, ..., $\bar{t}_n = \bar{t}$ are the bounds of these intervals. Evaluation of the integrals over $\bar{\tau}$ in each interval $\bar{t}_{k-1} - \bar{t}_k$ with allowance for the singularity of Green's function is carried out as follows:

$$\int_{\bar{t}_{k-1}}^{\bar{t}_k} \bar{q}\,(\theta_0)\,\frac{d\bar{\tau}}{\sqrt{\bar{t}_i - \bar{\tau}}} = \left[\bar{q}\,(\theta_{k-1}) + \bar{q}\,(\theta_k)\right]\left(\sqrt{\bar{t}_i - \bar{t}_{k-1}} - \sqrt{\bar{t}_i - \bar{t}_k}\right),$$

$$\int_{\bar{t}_{k-1}}^{\bar{t}_k} \bar{q}\,(\theta_0)\exp\left[-\frac{\bar{x}_j^2}{4\pi\,(\bar{t}_i - \bar{\tau})}\right]\frac{d\bar{\tau}}{\sqrt{\bar{t}_i - \bar{\tau}}} = \left[\bar{q}\,(\theta_{0k-1}) + \bar{q}\,(\theta_{0k})\right] f_0\,(j, i, k)\,.$$

The nonlinear term is computed as follows. The function $\bar{F}(\theta)$ is assumed to be time-independent and equal to $\bar{F}(\theta(\bar{t}_{k-1}, \xi))$ in each interval $\bar{t}_{k-1} - \bar{t}_k$. With such an approach, the relative error of integration with respect to $\bar{\tau}$ is no higher than $\left|\dfrac{\partial F}{\partial\theta}\,\dfrac{\theta(t_{k-1}, \xi) - \theta(t_k, \xi)}{\theta_0}\right|\bar{t}$. The integral over $\bar{\xi}$ is evaluated by the trapezium method. Finally, the double integral is reduced to double summation:

$$\int_{\bar{t}_{k-1}}^{\bar{t}_k}\int_0^{\infty} \bar{F}\,(\theta)\exp\left[-\frac{\bar{\xi}^2}{4\pi\,(\bar{t} - \tau)}\right]\frac{d\bar{\xi}d\bar{\tau}}{\sqrt{\bar{t} - \bar{\tau}}} = \sum_{k=2}^{i}\sum_{m=2}^{N_{k-1}} \bar{F}_{k-1,m}\left[f_0\,(m-1, i, k) + f_0\,(m, i, k)\right]\Delta\,.$$

Here we have introduced the following notation: $f_0(j, i, k) = \int_{\bar{t}_{k-1}}^{\bar{t}_k} \exp\left[-\dfrac{\bar{x}_j^2}{4\pi(\bar{t}_i - \bar{\tau})}\right]\dfrac{d\bar{\tau}}{2\sqrt{\bar{t}_i - \bar{\tau}}}$, $\bar{x}_k = (k-1)\Delta$, $\theta_{i0} = \theta_0(\bar{t}_i)$, and $\bar{F}_{k-1,m} = \bar{F}(\theta(\bar{t}_{k-1}, \bar{x}_m))$; Δ is the step of numerical integration with respect to the dimensionless coordinate $\bar{\xi}$. The upper limit of the sum N_{k-1} is selected so that the inequality $\theta(\bar{t}_{k-1}, \bar{x}_N) - 1 \le \varepsilon$ (ε is the small quantity taken to be 10^{-3} in the calculations) is fulfilled at the point $x_N = (N_{k-1} - 1)\Delta$.

The corresponding transformations for calculation of the temperature on the surface $\theta_0(\bar{t})$ result in the following system of recurrence relations:

$$\theta_{02} + \sqrt{\delta_2}\,\bar{q}_c\,(\theta_{02}) = 1 - \sqrt{\delta_2}\,\bar{q}_c\,(\theta_{01}) + B_{21}\,;\quad \theta_{0i} + \sqrt{\delta_i}\,\bar{q}_c\,(\theta_{0i}) = 1 - A_i + B_{i1}\,,\quad i = 3, ..., n\,, \tag{13}$$

whereas for calculation of the temperature inside the rod $\theta(\bar{x}, \bar{t})$, we obtain

$$\theta\,(\bar{x}_j, \bar{t}_i) = 1 - A_{ij} + B_{ij}\,. \tag{14}$$

Here we have introduced the following notation:

$$A_i = \bar{q}_c\,(\theta_1)\,d_{i1} + \sum_{k=2}^{i-1} \bar{q}_c\,(\theta_k)\,d_{ik}\,;\quad d_{i1} = \sqrt{\bar{t}_i} - \sqrt{\bar{t}_i - \bar{t}_2}\,;\quad d_{ik} = \sqrt{\bar{t}_i - \bar{t}_{k-1}} - \sqrt{\bar{t}_i - \bar{t}_{k+1}}\,;\quad \delta_i = t_i - t_{i-1}\,;$$

$$A_{ij} = f_0\,(j, i, 2)\,\bar{q}_c\,(\theta_1) + f_0\,(j, i, i)\,\bar{q}_c\,(\theta_i) + \sum_{k=2}^{i-1}\left[f_0\,(j, i, k) + f_0\,(j, i, k+1)\right]\bar{q}_c\,(\theta_{0k})\,;$$

$$B_{ij} = \sum_{k=2}^{i}\sum_{m=2}^{N_{k-1}} \bar{F}_{k-1,m}\left[f_1\,(j, m-1, i, k) + f_1\,(j, m, i, k)\right]\Delta\,;$$

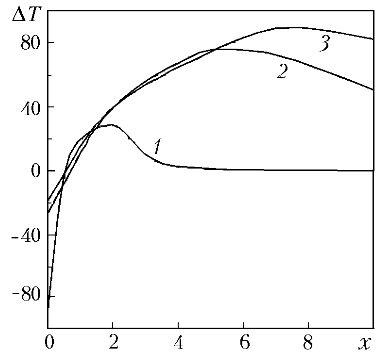

Fig. 1. Distribution of the difference of the calculated temperatures with variable and constant thermophysical properties: 1) t = 2, 2) 42, and 3) 70 sec. ΔT, K; x, mm.

$$f_1(j, m, i, k) = \int_{\bar{t}_{k-1}}^{\bar{t}_k} \left\{ \exp\left[-\frac{(\bar{x}_j - \bar{x}_m)^2}{4\pi(\bar{t}_i - \bar{\tau})} \right] + \exp\left[-\frac{(\bar{x}_j + \bar{x}_m)^2}{4\pi(\bar{t}_i - \bar{\tau})} \right] \right\} \frac{d\bar{\tau}}{4\sqrt{\bar{t}_i - \bar{\tau}}} .$$

The process of solution of Eqs. (13) and (14) is as follows. The initial condition $\theta(\bar{x})\big|_{t=0} = 1$ yields that $\bar{F}_{1j} = 0$. Substituting these values of $\bar{F}$ into Eq. (13), we find θ_{02}. Next, using (14) we determine the $\theta(\bar{t}_2, \bar{x})$ values, which are subsequently used in (13) in finding θ_{03}, etc. Such a procedure is continued until $\theta(\bar{t}_n, \bar{x})$ is found.

Example of Calculation and Discussion of the Results. We calculated, as an example, the temperature field in heating of a rod made of quartz glass ceramics (doped with chromium oxide which is used as a heatproof spacecraft material) and insulated on the lateral sides by an air flow at a temperature of 6000 K with a heat-transfer coefficient α = 470 W/(m^2·K). The thermophysical properties of the material [5] were as follows:

$$\lambda = (1.22 + 2.4 \cdot 10^{-4} \Delta T) \ \text{W/(m·K)} , \quad \Delta T = T - T_{\text{in}} , \quad T_{\text{in}} = 300 \ \text{K} ;$$

$$c = \{1.5 + 0.55 \,[1 - \exp(-1.9 \cdot 10^{-3} \Delta T)]\} \cdot 10^3 \ \text{J/(kg·K)} ; \ \rho = 2 \cdot 10^3 \ \text{kg/m}^3 , \quad \varepsilon_{\text{r}} = 0.8 . \tag{15}$$

The maximum heating time was 70 sec; the maximum depth of pronounced warmup was ~3 cm. The calculation by the described method was compared to the exact solution for constant thermophysical properties in the absence of radiation from the surface:

$$\frac{T - T_{\text{in}}}{T_{\text{g}} - T_{\text{in}}} = \text{erfc}\left(\frac{x}{2\sqrt{a_{\text{in}} t}} \right) - \text{erfc}\left(\frac{x}{2\sqrt{a_{\text{in}} t}} + \alpha \sqrt{\frac{t}{\lambda_{\text{in}} \rho c_{\text{in}}}} \right) \exp\left(\frac{\alpha x}{\lambda_{\text{in}}} + \frac{\alpha^2 t}{\lambda_{\text{in}} \rho c_{\text{in}}} \right) .$$

As the calculations showed, for t = 70 sec, satisfactory accuracy was attained even for n = 85 and Δ = 1 mm. The bounds of the time intervals were computed from the formula $t_i = 70\,(1 + \exp(i/10))/(1 + \exp(85/10))$. The maximum error of numerical calculation was observed in the initial step of heating (~2 sec) and amounted to ~1.3% at a depth of 1–2 mm. Subsequently it became lower and amounted to as low as 0.05–0.1% at t = 70 sec.

This method of calculation was compared to the finite-element numerical method. Thus, in [6], the problem on heating, for t = 0.03 sec, of a four-layer material with $a_{\text{in}} \approx 10^{-5}$ m^2/sec and a thickness of 15 cm by the heat flux with $q_{\text{c}} \approx 10^6$ W/m^2 was solved. It took 22,077 sec to solve it on a Pentium II 400 computer with a 128-Mb RAM by the finite-element method. Solution of the problem considered above by the numerical-analytical technique in a Pentium II 470 computer with a 192-Mb RAM took 7280 sec. Thus, taking into account different time and space intervals of these problems, we may infer that the numerical-analytical technique allows a reduction of more than 10^3 times in the calculation time compared to the finite-element method.

The influence of nonlinearity on the temperature profiles can be tracked by analyzing Fig. 1 which shows the distribution of the difference of the calculated temperatures ΔT with variable and constant thermophysical properties determined from formulas (15) at $T = T_{in} = 300$ K with allowance for the radiation from the surface for three instants of time. The behavior of the presented curves is quite unusual. Increase in the thermal conductivity and the heat capacity at high temperatures, i.e., near the heated surface, leads to a reduction in the temperature on the surface and in the nearby layers. The heat flux into the body increases due to both the growth in the difference of the gas and surface temperatures and the decrease in the radiant flux. This increase is compensated by the growth in the difference in temperatures in the deep layers of the material; the maximum, increasing with time, moves deep into the body.

Conclusions. Reduction of the nonlinear unsteady heat-conduction equation to a linear form with virtual heat sources enabled us to use the method of Green's functions in the process of its numerical solution. This technique made it possible to substantially reduce the expenditure of calculation time compared to the standard finite-element method.

NOTATION

a, thermal diffusivity, m^2/sec; a_{in}, thermal diffusivity with the initial conditions, m^2/sec; A, B, d, f, and δ, auxiliary functions in solving the system of Eqs. (13) and (14); c, heat capacity, J/kg; c_{in}, heat capacity with the initial conditions, J/(kg·K); F, function of virtual heat sources (sinks), K/sec; G and G_0, Green's functions, m^{-1}; n, number of time intervals; q_c, heat flux on the body's surface, W/m^2; q_r, radiant heat flux, W/m^2; T, body's temperature, K; T_{in}, body's initial temperature, K; T_0, surface temperature, K; T_g, gas temperature, K; t, time, sec; t_*, time scale, sec; x, coordinate, m; x_*, linear scale, m; α, coefficient of heat transfer to the surface, W/(m^2·K); β, radiation parameter of the surface; ε, small quantity; ε_r, emissivity factor of the surface; λ, thermal conductivity, W/(m·K); λ_{in}, thermal conductivity with the initial conditions, W/(m·K); ξ, auxiliary coordinate, m; ρ, density, kg/m^3; θ, dimensionless temperature; σ, Stefan–Boltzmann constant, W/(m^2·K^4); τ, auxiliary time. Superscripts and subscripts: in, initial; c, convective; g, gas; i and k, Nos. of time intervals; j and m, Nos. of coordinates of points; r, radiation; $^-$, dimensionless quantity.

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